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(2+1)-Dimensional Fermion Determinant in a Constant Background Field at Finite Temperature and Density

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We evaluate exactly both the nonrelativistic and relativistic fermion determinant in 2+1 dimensions in a constant background field at finite temperature. The effect of finite chemical potential is also considered. In both cases, the systems are decoupled into an infinite number of 1+1 fermions by Fourier transformation in the β-variable. The total effective actions demonstrate nonextensiveness in the β-dimension.

1. INTRODUCTION

Thanks to the exotic mathematical structure and the possible relevence to condensed matter physics in two space dimensions, Chern–Simons (CS) models have drawn much attention in the past decade (Jackiw, 1985; Randjbar-Daemi *et al.*, 1990). The CS term can be either put in by hand, or more naturaly, induced by fermion degrees, as a part of the original (effective) lagrangian. Two properties of the CS action are fundamental. One is that it is odd under parity transform because of the presence of three-dimensional Levi–Civita symbol. The other is that it is invariant under *small* gauge transforms while noninvariant under *large* gauge transforms (those not to be continuously deformed to unity and thus carrying nontrivial winding numbers) (Deser *et al.*, 1982a,b). In the free space–time whose topology is trivial, the homotopy group π_3 is trivial in the Abelian case. But there may be nontrivial large gauge transformations if the gauge fields are subject to nontrivial boundary conditions (for a more recent discussion see Deser *et al.*, 1997, 1998). In general, if there exists nontrivial π_3,

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the quantum theory is consistent only if the CS parameters are quantized. There then arises a problem: What happens to the quantized parameters by quantum corrections? In zero temperatures, the induced CS term is well understood (Coleman, 1985; Pisarski, 1985; Redlich, 1984a,b; Rothe, 1993; Witten, 1982). But at finite temperatures, it was argued (Pisarski, 1987) that the coefficient of the CS term in the effective action for the gauge field should remain unchanged. Yet, a naive perturbative calculation that mimics that at zero temperature leads to a CS term with a parameter continuously dependent on the temperature (Niemi, 1985; Niemi and Semenoff, 1985; Pisarski, 1987; Redlich and Wijewardhana, 1985). Therefore, the behavior under gauge transforms seems to be temperature dependent. The problem of quantum corrections to the CS coefficient induced by fermions at finite temperature was reexamined in (Bralic *et al.*, 1996 and Cabra *et al.*, 1996), where it was concluded that, on gauge invariance grounds and in perturbation theory, the effective action for the gauge field cannot contain a smoothly renormalized CS coefficient at nonzero temperatures. Obviously, it is neccessary to obtain some exact result to reconcile the contradiction. As a toy model, the effective action of a 0+1 analog of the 2+1 CS system was exactly calculated (Dunne *et al.*, 1997). It shows that in the analog, the exact finite temperature effective action, which is nonextensive in temperature, has a well-defined behavior under a large gauge transformation, *independent of the temperature*, even though at any given finite order of a perturbation expansion, there is a temperature dependence. So it implies that the discussions of the gauge invariance of finite temperature effective actions and induced CS terms in higher dimensions requires consideration of the full perturbation series. Conversely, no sensible conclusions may be drawn by considering only the first finite number of terms in the expansion. The course of being exactly calculable is that the gauge field can be made constant by gauge transformations. Employing this trick, Fosco *et al.* calculated exactly the parity breaking part of the fermion determinant in 2+1 dimensions with a particular background gauge field for both Abelian and non-Abelian cases (Fosco *et al.*, 1997a,b), and the result agrees with that from the ζ-function method (Deser *et al.*, 1997, 1998). More general background gauge fields were also considered (Aitchison and Fasco, 1998). All these works show that (restricted to that particular ad hoc configuration) gauge invariance of the effective action is respected even when large gauge transformations are considered. It is now clear that the effective action induced by the fermion determinant is in general a nonextensive quantity in space–time/temperature and this feature enables the effective action preserve gauge invariance.

In the nonrelativistic case, the effective action induced by the 2+1 fermion determinant was studied in (Randjbar-Daemi *et al.*, 1990) perturbatively in order to investigate the possible relevance between CS theory and superconductivity, at both zero and finite temperatures. Since the determinant can not be evaluated exactly for general background gauge fields, (Neagu and Schakel, 1993) and Andersen and

Haugset[4] considered the case that the gauge field is that of a constant magnetic field and discussed the induced quantum numbers and de Hass–van Alphen effect. The difference between the perturbative (loop) calculations and the rigorous results in this special case is demonstrative.

The effect of finite chemical potential should be taken into account whenever discussing the statistical physics of a grand canonical ensemble. It was shown that in 1+1 dimensions, the nonzero chemical potential may contribute a nontrivial phase factor to the partition function (Alvarez-Estrada and Nicola, 1998). The problem for an arbitrary background in 2+1 dimensions was tackled perturbatively by Sissakian and coworkers.[5] As ususal, gauge transform property of the effective action suffers some temperature dependence. Using the same technique as in (Fosco *et al.*, 1997a), the effect on the parity-odd part of nonzero chemical potential is considered in (Feng and Zhu, 1999) but the parity-even part can not be obtained exactly for the background therein. Therefore, it is worthwhile considering the problem by exact computation with some particular background. This is the topic of this paper. The layout of this paper is as follows: In Sections 2 and 3 we exactly evaluate the nonrelativistic and relativistic fermion determinant at finite temperature and finite density in a constant magnetic field. Section 4 is devoted to conclusionary discussions.

2. THE NONRELATIVISTIC CASE

The fermion Lagrangian is (Feng *et al.*, 1997)

$$\mathcal{L} = \psi^\dagger i D_0 \psi - \frac{1}{2m}\psi^\dagger \not{\mathbf{D}}^2 \psi \tag{1}$$

where $\not{\mathbf{D}} = \gamma^i D_i$, $i = 1, 2 \cdot D_\mu = \partial_\mu + ieA_\mu$, $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^2 = i\sigma_1 \cdot e = -|e|$, and $\sigma_{1,2,3}$ are the usual three Pauli matrices. We choose representation of gamma matrices so that it gives the correct sign of the Zeeman energy. It can be calculated directly that

$$\frac{1}{2m}\not{\mathbf{D}}^2 = \frac{1}{2m}\left(D_i D^i + \frac{1}{4}[\gamma^i, \gamma^j] i\, eF_{ij}\right) = \frac{1}{2m}(-\mathbf{D}^2) - g_s \mu_B B s_z \tag{2}$$

where $\mu_B = e/2m$, and $g_s = 2$ is the electron g-factor for spin. We incorporate an external field b to discuss the spin. The Euclidean action at finite temperature and finite density reads then

$$\mathcal{L}_E = \psi^\dagger \left[D_\tau - \frac{1}{2m}\not{\mathbf{D}}^2 - b s_z + \mu \right] \psi \tag{3}$$

[4] J. O. Andersen and T. Haugset, hep-th/9410084.

[5] A. N. Sissakian, O. Yu. Shevchenko, and S. B. Solganik, *Topological Effects in Medium*, hep-th/9806047.

The effective action Γ is given by definition

$$e^{-\Gamma} = \int_{A.B.C.} \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left\{ -\int_0^\beta d\tau \int d^2\mathbf{x} \left[\psi^\dagger D_\tau \psi - \frac{1}{2m}\psi^\dagger \not{\mathbf{D}}^2 \psi \right.\right. \\ \left.\left. + \mu\psi^\dagger\psi - b\psi^\dagger s_z \psi \right] \right\} \tag{4}$$

where the $A.B.C.$ implies that the functional integral over the fermion fields is implemented with antiperiodic boundary conditions. Once Γ is known, the induced particle number and spin are provided by

$$\langle N \rangle = \int d^2\mathbf{x} \langle \psi^\dagger \psi \rangle = -\frac{1}{\beta}\frac{\partial}{\partial \mu}\Gamma_{b=0} \tag{5}$$

$$\int d^2\mathbf{x} \langle \psi^\dagger s_z \psi \rangle = -\frac{1}{\beta}\frac{\partial}{\partial b}\Gamma_{b=0} \tag{6}$$

where we have used the fact that the correlation functions in (5) and (6) are actually β-independent.

Since the exact evaluation of the functional integration in (4) in general is beyond our ability so far, we first consider those backgrounds of the following space–time dependence as in (Fosco *et al.*, 1997a).

$$A_\tau = A_\tau(\tau), \qquad A_i = A_i(\mathbf{x}) \tag{7}$$

Then the gauge field can be rendered constant in "time" τ by gauge transformations. The time component A_τ will be its average value $\tilde{A}_\tau = \beta^{-1}\int_0^\beta d\tau A_\tau(\tau)$, which can not be transformed away by *local* transformations. In this gauge, we can employ the Fourier transformation

$$\psi(\tau, \mathbf{x}) = \beta^{-1/2}\sum_{-\infty}^{+\infty} \psi_n(\mathbf{x})\, e^{i\omega_n \tau}, \qquad \psi^\dagger(\tau, \mathbf{x}) = \beta^{-1/2}\sum_{-\infty}^{+\infty} \psi_n^\dagger(\mathbf{x})\, e^{-i\omega_n \tau} \tag{8}$$

where $\omega_n = \frac{(2n+1)\pi}{\beta}$, to decouple the system as a sum of an infinite number of fermions in 1+1 dimensions.

$$e^{-\Gamma} = \prod_n \int \mathcal{D}\psi_n^\dagger(\mathbf{x})\, \mathcal{D}\psi_n(\mathbf{x}) \, \exp\left\{ -\int d^2\mathbf{x}\, \psi_n^\dagger(\mathbf{x}) \right. \\ \left. \times \left[(i\omega_n + ie\tilde{A}_\tau) - \frac{1}{2m}(-\mathbf{D}^2) - b' s_z + \mu \right] \psi_n(\mathbf{x}) \right\} \tag{9}$$

($b' = b - g_s \mu_B B$) where we have used the transformation of the functional measure

$$\mathcal{D}\psi^\dagger(\tau, \mathbf{x})\mathcal{D}\psi(\tau, \mathbf{x}) = \prod_n \mathcal{D}\psi_n^\dagger(\mathbf{x})\mathcal{D}\psi_n(\mathbf{x}) \tag{10}$$

which can be easily proved from the orthonormality of the basis $\{\beta^{-1/2} e^{i\omega_n \tau}\}$ in the Fourier transformation. It can be seen easily that once the eigenvalues of the operator $\frac{1}{2m}(-D^2) + b' s_z$ are known, the functional integration can be accomplished readily. Unfortunately, this is impossible for general gauge field backgrounds, even for the restricted class (7). Therefore, we need to make further restrictions. The simplest case is that the magnetic field F_{ij} is constant, $F_{12} = B$, and the corresponding gauge potential can be chosen in the gauge $\mathbf{A} = (-By, 0)$. In this case, the eigenvalues of the operator $\frac{1}{2m}(-D^2) + b' s_z$ can be acquired from the solutions of the equation

$$\left\{ \frac{1}{2m} \left[(P_x + eBy)^2 + P_y^2 \right] + b' s_z \right\} \chi = \lambda \chi \tag{11}$$

where χ is a two-component spinor and $P_i = -i\partial_i$. The solutions to (11) are easy to find and the eigenvalues can be obtained from the well-known Landau levels, that is

$$\lambda_{l,s_z} = \left(l + \frac{1}{2} \right) \Omega + b' s_z \quad l = 0, 1, 2, \ldots; \;\; s_z = \pm\frac{1}{2}; \;\; \Omega = \frac{|eB|}{m} \tag{12}$$

These energy levels are highly degenerate with degeneracy $\frac{|eB|}{2\pi}$ per unit area, which must be taken into account when calculating the fermion determinant.

$$e^{-\Gamma} = \prod_n \mathrm{Det}\left[i\omega_n + ie\tilde{A}_\tau - \frac{1}{2m}(-\mathbf{D}^2) - b' + \mu \right] \tag{13}$$

There is one important point that deserves attention here. In the absence of external magnetic field, the Hamiltonian is just that of a free electron and the energy eigenvalue spectrum is continuous, which can not be regarded simply as the limit of the discrete spectrum for vanishing external field. Since the numerator is

$$\begin{aligned} &\mathrm{Det}\left[i\omega_n + ie\tilde{A}_\tau - \frac{1}{2m}(-\mathbf{D}^2) - b' s_z + \mu \right] \\ &= \left\{ \prod_{l=0}^{\infty} \prod_{s_z = \pm\frac{1}{2}} [i\omega_n - E_{l\pm} + \mu] \right\}^{\frac{|eB|}{2\pi}} \end{aligned} \tag{14}$$

$$E_{l\pm} = ie\tilde{A}_\tau + \left(l + \frac{1}{2} \right) \Omega \pm \frac{b'}{2} \tag{15}$$

we have

$$-\Gamma = \frac{|eB|}{2\pi} \sum_l \sum_n [\ln(i\omega_n - E_{l+} + \mu) + \ln(i\omega_n - E_{l-} + \mu)] \tag{16}$$

Using the formula for fermion (Fetter and Walecka, 1971)

$$\sum_n \frac{1}{i\omega_n - x} = \frac{\beta}{e^{\beta x} + 1} \tag{17}$$

We have then the expectation values of the spin-up and spin-down electrons per unit area

$$N_\pm = \frac{|eB|}{2\pi} \sum_l \frac{1}{e^{\beta(E_{l\pm} - \mu)} + 1} \tag{18}$$

$$\langle s_z \rangle = \frac{1}{2}(N_+ - N_-) \tag{19}$$

$$M_z = g_s \mu_B \langle s_z \rangle \tag{20}$$

At zero temperature, these results coincide with those of (Neagu and Schakel, 1993).

3. THE RELATIVISTIC CASE

The Lagrangian of the fermion is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \not{D}_\mu - m)\psi \tag{21}$$

There are two inequivalent representations of the γ-matrices in three dimensions: $\gamma^\mu = (\sigma_3, i\sigma_2, i\sigma_1)$ and $\gamma^\mu = (-\sigma_3, -i\sigma_2, -i\sigma_3)$. We choose the first. As usual, the total effective action $\Gamma(A, m, \mu)$ at finite temperature is defined as

$$e^{-\Gamma(A,m,\mu)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\int_0^\beta d\tau \int d^2x\, \bar{\psi}(\not{\partial} + ie\not{A} + m - \mu\gamma^3)\psi\right] \tag{22}$$

where we are using Euclidean Dirac matrices in the representation $\gamma_\mu = (\sigma_3, \sigma_2, \sigma_1)$, and β is the inverse temperature. It makes no difference whether the indices are lower or upper. The label 3 refers actually to the Euclidean time component. The fermion fields are subject to antiperiodic boundary conditions while the gauge field are periodic. Under parity transformation,

$$\begin{aligned} &x^1 \to -x^1, x^2 \to x^2, x^3 \to x^3; \quad \psi \to \gamma^1\psi, \bar{\psi} \to -\bar{\psi}\gamma^1; \\ &A^1 \to -A^1, A^2 \to A^2, A^3 \to A^3 \end{aligned} \tag{23}$$

(γ matrices are kept intact). So only the mass term varies under the parity transformation. As in (Fosco *et al.*, 1997a), the parity-odd part is defined as

$$2\Gamma(A, m, \mu)_{\text{odd}} = \Gamma(A, m, \mu) - \Gamma(A, -m, \mu) \tag{24}$$

It is not an easy task to calculate (22) for general configuration of the gauge field. A particular class of configurations of A for which (22) can be exactly computed is that defined by (7). This class of gauge fields shares the same feature as in the $0 + 1$ dimensions in (Dunne *et al.,* 1997): the time dependence of the time component can be erased by gauge transformations. Therefore, the Euclidean action can be decoupled as a sum of an infinite 1+1 actions

$$e^{-\Gamma(A,m,\mu)} = \int \mathcal{D}\psi_n(x)\mathcal{D}\bar{\psi}_n(x)\, \exp\left\{-\frac{1}{\beta}\sum_{-\infty}^{+\infty}\int d^2x\, \bar{\psi}_n(x) \times [\not{d} + m + i\gamma^3(\omega_n + e\tilde{A}_3) - \mu\gamma^3]\psi_n(x)\right\} \tag{25}$$

where $\not{d} = \gamma_j(\partial_j + i\, eA_j)$ is the 1+1 Dirac operator and $\tilde{A}_3$ is the mean value of $A_3(\tau)$. It is seen that the chemical potential in 2+1 dimensions plays the role of a chiral potential in 1+1 dimensions. Let us introduce Ω_n for convenience, $\Omega_n = \omega_n + e\tilde{A}_3$. Since

$$m + i\gamma^3\Omega_n - \mu\gamma^3 = \rho_n\, e^{i\gamma_3\phi_n} \tag{26}$$

where

$$e^{2i\phi_n} = \frac{m - \mu + i\Omega_n}{m + \mu - i\Omega_n} \tag{27}$$

and

$$\rho_n = \sqrt{(m + \mu - i\Omega_n)(m - \mu + i\Omega_n)} \tag{28}$$

we have therefore

$$\mathrm{Det}(\not{\partial} + ie\not{A} + m - \mu\gamma^3) = \prod_{n=-\infty}^{+\infty} \mathrm{Det}[\not{d} + \rho_n e^{i\gamma_3\phi_n}] \tag{29}$$

Explicitly, the 1+1 determinant for a given mode is a functional integral over 1+1 fermions

$$\mathrm{Det}[\not{d} + \rho_n e^{i\gamma_3\phi_n}] = \int \mathcal{D}\chi_n\mathcal{D}\bar{\chi}_n\, \exp\left\{-\int d^2x\, \bar{\chi}_n(x)(\not{d} + \rho_n e^{i\gamma_3\phi_n})\chi_n(x)\right\} \tag{30}$$

After implementing a chiral rotation whose Jacobian is well-known (the Fujikawa method applies also to complex chiral parameters), we obtain

$$\mathrm{Det}[\not{d} + m + i\gamma^3(\omega_n + e\tilde{A}_3) - \mu\gamma^3] = J_n\, \mathrm{Det}[\not{d} + \rho_n] \tag{31}$$

where

$$J_n = \exp\left(-i\,\frac{e\phi_n}{2\pi}\int d^2x\,\epsilon^{jk}\partial_j A_k\right) \tag{32}$$

Note that the chiral anomalies, or the Jacobian J, depends on the boundary conditions as well. If the system is defined on a torus and the fields are subject to periodic boundary conditions, for instance $A_j(x, y) = A_j(x + L_x, y)$, $A_j(x, y) = A_j(x, y + L_y)$, the trace of γ_5 in (Abdalla *et al.*, 1991) is taken over the discrete complete set instead of the continuous plane waves. Thus the momentum integral $\int \frac{d^2k}{(2\pi)^2} e^{-k^2} = \frac{1}{4\pi}$ should be replaced by $\frac{1}{L_xL_y}\sum_{n_1,n_2}\exp[-(\frac{2\pi}{L_x}n_1)^2 - (\frac{2\pi}{L_2}n_2)^2]$. Using the formula $\sum_{n=\infty}^{+\infty} e^{-\pi z n^2} = \frac{1}{\sqrt{z}}\sum_{n=-\infty}^{+\infty} e^{-\frac{\pi}{z}n^2}$ (Burckel, 1979) which holds for any complex z with $Re\, z > 0$, we have

$$\frac{1}{L_xL_y}\sum_{n_1,n_2}\exp\left[-\left(\frac{2\pi}{L_x}n_1\right)^2 - \left(\frac{2\pi}{L_2}n_2\right)^2\right] = \theta(L_x)\theta(L_y) \tag{33}$$

where $\theta(L) = \frac{1}{\sqrt{4\pi}}\sum_{n=-\infty}^{+\infty} e^{-\frac{L^2}{4}n^2}$. In this case, (32) should be replaced by

$$J_n = \exp\left(-2ie\phi_n\theta(L_x)\theta(L_y)\int_{L_x\times L_y} d^2x\,\epsilon^{jk}\partial_j A_k\right) \tag{34}$$

In the following, we only concentrate on the infinite space case since the conclusion on a torus can be obtained by a trivial substitution. Fortunately, we also have $\rho_n(m) = \rho_n(-m)$ for finite chemical potential. Thus we have immediately

$$\Gamma_{\text{odd}} = -\sum_{n=-\infty}^{+\infty}\ln J_n = i\,\frac{e}{2\pi}\sum_{n=-\infty}^{+\infty}\phi_n\int d^2x\,\epsilon^{jk}\partial_j A_k \tag{35}$$

To calculate $\sum_{n=-\infty}^{+\infty}\phi_n$, we need to compute $\prod_{n=-\infty}^{+\infty}\frac{m-\mu+i\Omega_n}{m+\mu-i\Omega_n}$. Using the formula $\prod_{n=1,2,3,\ldots}[1 - \frac{4a^2}{(2n-1)^2}] = \cos\pi a$ as in Jackiw (1985), we have ($a = e\tilde{A}_3$)

$$\prod_{n=-\infty}^{+\infty} e^{2i\phi_n} = \prod_{n=-\infty}^{+\infty}\frac{m-\mu+i\Omega_n}{m+\mu-i\Omega_n} = \frac{\text{ch}\frac{\beta}{2}(m-\mu) + i\,\text{sh}\frac{\beta}{2}(m-\mu)\text{tg}\frac{\beta a}{2}}{\text{ch}\frac{\beta}{2}(m+\mu) - i\,\text{sh}\frac{\beta}{2}(m+\mu)\text{tg}\frac{\beta a}{2}} \tag{36}$$

Therefore

$$\Gamma_{\text{odd}} = \frac{e}{4\pi}\ln\left[\frac{\text{ch}\frac{\beta}{2}(m-\mu) + i\,\text{sh}\frac{\beta}{2}(m-\mu)\text{tg}\frac{\beta a}{2}}{\text{ch}\frac{\beta}{2}(m+\mu) - i\,\text{sh}\frac{\beta}{2}(m+\mu)\text{tg}\frac{\beta a}{2}}\right]\int d^2x\,\epsilon^{jk}\partial_j A_k \tag{37}$$

which is quite different from the perturbative conclusion in (Neagu and Schakel, 1993) (The formula (97) there is for an arbitrary background).

Now the low temperature limit can be obtained. It will depend on the relationship between m and μ.

(i) If $m > \mu, m + \mu > 0$

$$\lim_{\beta\to\infty} \Gamma_{\text{odd}} = \frac{e}{4\pi}\beta(ia - \mu)\int d^2x\, \epsilon^{jk}\partial_j A_k \tag{38}$$

(ii) If $m - \mu > 0, m + \mu < 0$,

$$\lim_{\beta\to\infty} \Gamma_{\text{odd}} = \frac{e}{4\pi}\beta m\int d^2x\, \epsilon^{jk}\partial_j A_k \tag{39}$$

(iii) If $m < \mu, m + \mu > 0$

$$\lim_{\beta\to\infty} \Gamma_{\text{odd}} = \frac{e}{4\pi}(-\beta m)\int d^2x\, \epsilon^{jk}\partial_j A_k \tag{40}$$

(iv) If $m < \mu, m + \mu < 0$

$$\lim_{\beta\to\infty} \Gamma_{\text{odd}} = \frac{e}{4\pi}\beta(\mu - ia)\int d^2x\, \epsilon^{jk}\partial_j A_k \tag{41}$$

(v) If $m = \mu$

$$\lim_{\beta\to\infty} \Gamma_{\text{odd}} = \frac{e}{4\pi}\left(-\beta m + i\frac{\beta a}{2}\right)\ln\cos\frac{\beta a}{2}\int d^2x\, \epsilon^{jk}\partial_j A_k \tag{42}$$

It vanishes in the high temperature limit. It is obvious that the low temperature is very sensitive to the values of m and μ, as agrees with the results perturbatively obtained (Neagu and Schakel, 1993).

Since in the large-m limit (or in the low-density limit), the parity-odd part dominates over the effective action, and the particle number in the ensemble is $\langle N\rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z(\beta,\mu)$, we have from the limits (38) and (41) that the flux should be quantized,[6]

$$\Phi = \langle N\rangle\frac{4\pi\hbar}{e} \tag{43}$$

which implies that each particle carries flux $\frac{4\pi\hbar}{e}$ and thus should be of fractional spin $S_\otimes = \frac{1}{4}$. This is in accordance with the conclusion in (Neagu and Schakel, 1993).

The next thing is to evaluate $\text{Det}(\not{d} + \rho_n)$. We have to calculate the eigenvalues of the operator $\not{d} + \rho_n$ for this purpose, i.e., to solve the equation

$$(\not{d} + \rho_n)\psi = \lambda\psi \tag{44}$$

[6] The Eq (23) in (Feng and Zhu, 1999) should read $\Phi = \langle N\rangle\frac{4\pi\hbar}{e}$.

In general, it is impossible to solve it. So we confine ourselves to the background (7). It is easily seen that once the eigenvalues of $\not{d}$ are known, the eigenvalues λ can be obtained. We thus consider the problem

$$\not{d}\psi = a\psi \tag{45}$$

Since

$$\not{d}^2 = D_i D_i \frac{1}{4}[\gamma_j, \gamma_i][D_j, D_i] = D_i D_i + eB\sigma_3 \tag{46}$$

the eigenvalues a can be obtained from the well-known relativistic Landau levels (Jonhson and Lippmann, 1949).

$$a = \pm i\sqrt{2\left(l + \frac{1}{2}\right)|eB| - 2eBs_{\pm}} \tag{47}$$

with degeneracy $\frac{|eB|}{2\pi}$ per unit area. Accordingly, we have

$$\lambda_{l,s_{\pm}} = \rho_n \pm i\sqrt{2\left(l + \frac{1}{2}\right)|eB| - 2eBs_{\pm}} \tag{48}$$

Therefore, we have (we suppose $eB > 0$) for a unit area

$$\begin{aligned}
\mathrm{Det}(\not{d} + \rho_n) &= \frac{|eB|}{2\pi} \prod_{l=0}^{\infty} \left[\rho_n + i\sqrt{2\left(l + \frac{1}{2}\right)|eB| - eB}\right] \\
&\quad \times \left[\rho_n - i\sqrt{2\left(l + \frac{1}{2}\right)|eB| + eB}\right] \qquad (49) \\
&= \frac{|eB|}{2\pi} \rho_n \prod_{l=0}^{\infty} \left[\rho_n + i\sqrt{2(l+1)eB}\right]\left[\rho_n - i\sqrt{2(l+1)eB}\right] \qquad (50) \\
&= \frac{|eB|}{2\pi} \rho_n \prod_{l=0}^{\infty} (\rho^2 + 2(l+1)|eB|) \qquad (51)
\end{aligned}$$

Another way to evaluate it is to make use of the relation

$$\mathrm{Det}(\not{d} + \rho_n) = \mathrm{Det}[\sigma_3(\not{d} + \rho_n)\sigma_3] = \mathrm{Det}(-\not{d} + \rho_n) \tag{52}$$

from which one can deduce that

$$\mathrm{Det}(\not{d} + \rho_n) = \sqrt{\mathrm{Det}\left(-\not{d}^2 + \rho_n^2\right)} \tag{53}$$

The eigenvalue equation of $-\not{d}^2 + \rho_n^2$ is

$$\left(-D_i D_i - eB\sigma_3 + \rho_n^2\right)\psi = \nu\psi \tag{54}$$

Again from the Landau levels, we know that

$$\nu = 2eB(l + 1/2 - s_z) + \rho_n^2 \tag{55}$$

Therefore,

$$\text{Det}(\not{d} + \rho_n) = \frac{|eB|}{2\pi}\sqrt{\prod_{l=0}^{\infty}\left(2eBl + \rho_n^2\right)\left[2eB(l+1) + \rho_n^2\right]} \tag{56}$$

$$= \frac{|eB|}{2\pi}\rho_n \prod_{l=0}^{\infty}\left[2(l+1)eB + \rho_n^2\right] \tag{57}$$

which agrees with (51).

The total effective action is then

$$\Gamma = \Gamma_{\text{odd}} - \frac{|eB|}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\ln\rho_n + \sum_{l=0}^{\infty}\ln\left[\rho_n^2 + 2(l+1)eB\right]\right) \tag{58}$$

which is divergent. With this effective action, one can discuss the induced particle density and the spin of the system. But the expressions are not as simple as in the nonrelativistic case.

4. DISCUSSIONS

To conclude this paper, we make some discussions. For the background (7), the effective action can also be computed as the zero temperature case in (Neagu and Schakel, 1993). We here first seperate Γ into a parity-odd part and a parity-even part. Both calculations should be in accordance with each other. We know that in general at zero temperature, the functional determinant can be expanded in terms of the powers of $\frac{1}{m}$ (Deser and Redlich, 1988)

$$-i \ln \text{Det}(i\not{D} \pm m) = \pm W_{\text{CS}} + \frac{1}{24\pi m}\int d^3x\, F_{\mu\nu}F^{\mu\nu} + O\left(\frac{\partial^2}{m^2}\right) \tag{59}$$

Unfortunately, we can not make a direct comparison between (58) and (59) because of the sum $\sum_l$. Equation (58) can be written as

$$\Gamma = \Gamma_{\text{odd}} - \frac{|eB|}{2\pi}\cdot 3\sum_{n=-\infty}^{\infty}\ln\rho_n - \frac{|eB|}{2\pi}\sum_{l=0}^{\infty}\sum_{n=-\infty}^{\infty}\ln\left(1 + \frac{2(l+1)\,eB}{\rho_n^2}\right) \tag{60}$$

So the second term in (59) should correspond to the first term of the expansion of $\ln(1+x)$ of the third term in (60). In the case $\mu = \tilde{A}_3 = 0$, the sum over n can be accomplished using the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2+\theta^2} = \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{e^{\theta}+1}\right) \tag{61}$$

But the sum over l is troublesome.

Finally, we would like to mention that apart from the interests explained in the Introduction, there is another interest relevent to bosonization. If the fermion determinants can be calculated exactly, we may employ the duality-transformation approach (Burgess and Quevedo, 1994) to bosonize the fermion models as in (Feng *et al.*, 1998).

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